Part 1 Value

2 CHAPTER

How to Calculate Present Values

Companies invest in lots of things. Some are *tangible assets*—that is, assets you can kick, like factories, machinery, and offices. Others are *intangible assets,* such as patents or trademarks. In each case the company lays out some money now in the hope of receiving even more money later.

Individuals also make investments. For example, your college education may cost you $40,000 per year. That is an investment you hope will pay off in the form of a higher salary later in life. You are sowing now and expecting to reap later.

Companies pay for their investments by raising money and, in the process, assuming liabilities. For example, they may borrow money from a bank and promise to repay it with interest later. You also may have financed your investment in a college education by borrowing money that you plan to pay back out of that fat salary.

All these financial decisions require comparisons of cash payments at different dates. Will your future salary be sufficient to justify the current expenditure on college tuition? How much will you have to repay the bank if you borrow to finance your degree?

In this chapter we take the first steps toward understanding the relationship between the values of dollars today and dollars in the future. We start by looking at how funds invested at a specific interest rate will grow over time. We next ask how much you would need to invest today to produce a specified future sum of money, and we describe some shortcuts for working out the value of a series of cash payments.

The term *interest rate* sounds straightforward enough, but rates can be quoted in different ways. We, therefore, conclude the chapter by explaining the difference between the quoted rate and the true or effective interest rate.

Once you have learned how to value cash flows that occur at different points in time, we can move on in the next two chapters to look at how bonds and stocks are valued. After that we will tackle capital investment decisions at a practical level of detail.

For simplicity, every problem in this chapter is set out in dollars, but the concepts and calculations are identical in euros, yen, or any other currency.

2-1 Future Values and Present Values

Calculating Future Values

Money can be invested to earn interest. So, if you are offered the choice between $100 today and $100 next year, you naturally take the money now to get a year's interest. Financial managers make the same point when they say that money has a *time value* or when they quote the most basic principle of finance: *a dollar today is worth more than a dollar tomorrow.*

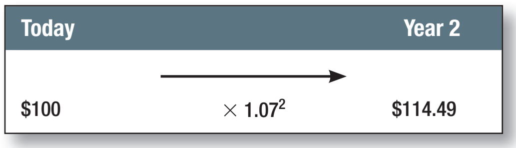
Suppose you invest \$100 in a bank account that pays interest of $r = 7\%$ a year. In the first year you will earn interest of $.07 \times \$100 = \$7$ and the value of your investment will grow to \$107:

$$\text{Value of investment after 1 year} = \$100 \times (1 + r) = 100 \times 1.07 = \$107$$

By investing, you give up the opportunity to spend \$100 today, but you gain the chance to spend \$107 next year.

If you leave your money in the bank for a second year, you earn interest of $.07 \times \$107 = \$7.49$ and your investment will grow to \$114.49:

$$\text{Value of investment after 2 years} = \$107 \times 1.07 = \$100 \times 1.07^2 = \$114.49$$



Notice that in the second year you earn interest on both your initial investment (\$100) and the previous year's interest (\$7). Thus your wealth grows at a *compound rate* and the interest that you earn is called **compound interest.**

If you invest your \$100 for t years, your investment will continue to grow at a 7% compound rate to $\$100 \times (1.07)^t$. For any interest rate r, the future value of your \$100 investment will be

$$\text{Future value of \$100} = \$100 \times (1 + r)^t$$

The higher the interest rate, the faster your savings will grow. Figure 2.1 shows that a few percentage points added to the interest rate can do wonders for your future wealth. For example, by the end of 20 years \$100 invested at 10% will grow to $\$100 \times (1.10)^{20} = \$672.75$. If it is invested at 5%, it will grow to only $\$100 \times (1.05)^{20} = \$265.33$.

Calculating Present Values

We have seen that \$100 invested for two years at 7% will grow to a future value of $100 \times 1.07^2 = \$114.49$. Let's turn this around and ask how much you need to invest *today* to

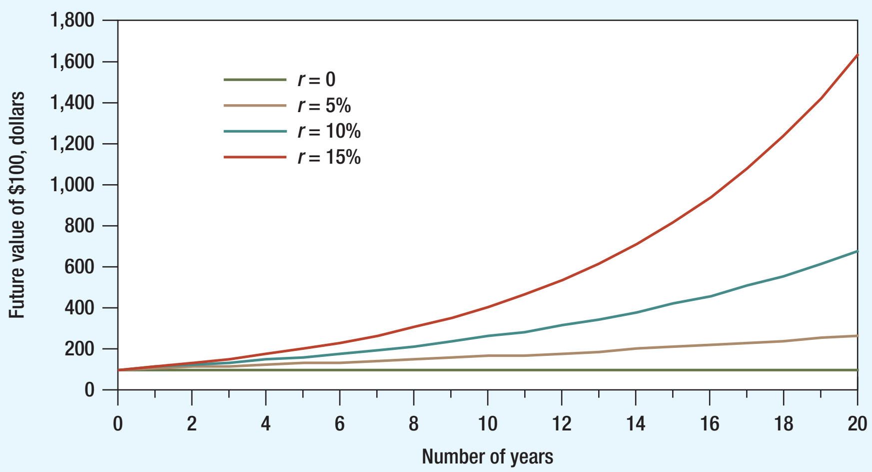


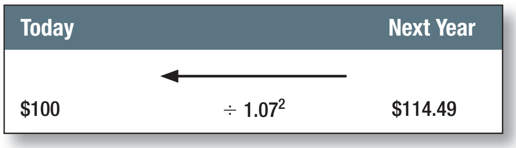
FIGURE 2.1

How an investment of \$100 grows with compound interest at different interest rates.

produce \$114.49 at the end of the second year. In other words, what is the **present value (PV)** of the \$114.49 payoff?

You already know that the answer is \$100. But, if you didn't know or you forgot, you can just run the future value calculation in reverse and divide the future payoff by $(1.07)^2$:

$$\text{Present value} = \text{PV} = \frac{\$114.49}{(1.07)^2} = \$100$$



In general, suppose that you will receive a cash flow of C_t dollars at the end of year t. The present value of this future payment is

$$\text{Present value} = \text{PV} = \frac{C_t}{(1 + r)^t}$$

The rate, r, in the formula is called the discount rate, and the present value is the discounted value of the cash flow, C_t. You sometimes see this present value formula written differently. Instead of *dividing* the future payment by $(1 + r)^t$, you can equally well *multiply* the payment by $1/(1 + r)^t$. The expression $1/(1 + r)^t$ is called the **discount factor.** It measures the present value of one dollar received in year t. For example, with an interest rate of 7% the two-year discount factor is

$$\text{DF}_2 = 1/(1.07)^2 = .8734$$

Investors are willing to pay \$.8734 today for delivery of \$1 at the end of two years. If each dollar received in year 2 is worth \$.8734 today, then the present value of your payment of \$114.49 in year 2 must be

$$\text{Present value} = \text{DF}_2 \times C_2 = .8734 \times 114.49 = \$100$$

The longer you have to wait for your money, the lower its present value. This is illustrated in Figure 2.2. Notice how small variations in the interest rate can have a powerful effect on the present value of distant cash flows. At an interest rate of 5%, a payment of \$100 in year 20 is worth \$37.69 today. If the interest rate increases to 10%, the value of the future payment falls by about 60% to \$14.86.

Valuing an Investment Opportunity

How do you decide whether an investment opportunity is worth undertaking? Suppose you own a small company that is contemplating construction of a suburban office block. The cost of buying the land and constructing the building is \$700,000. Your company has cash in the bank to finance construction. Your real-estate adviser forecasts a shortage of office space and predicts that you will be able to sell next year for \$800,000. For simplicity, we will assume initially that this \$800,000 is a sure thing.

The rate of return on this one-period project is easy to calculate. Divide the expected profit (\$800,000 − 700,000 = \$100,000) by the required investment (\$700,000). The result is 100,000/700,000 = .143, or 14.3%.

Figure 2.3 summarizes your choices. (Note the resemblance to Figure 1.2 in the last chapter.) You can invest in the project, or pay cash out to shareholders, who can invest on their own. We assume that they can earn a 7% profit by investing for one year in safe assets (U.S. Treasury debt securities, for example). Or they can invest in the stock market, which is risky but offers an average return of 12%.

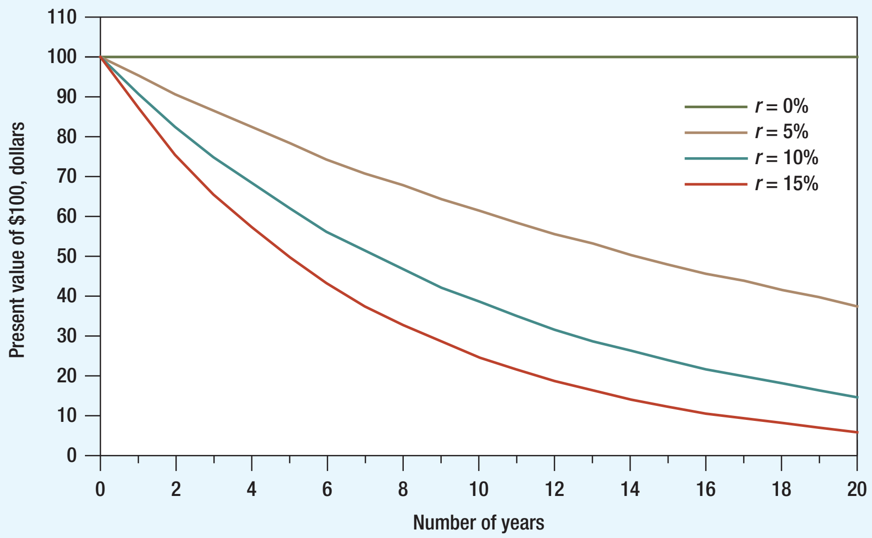


FIGURE 2.2

Present value of a future cash flow of $100. Notice that the longer you have to wait for your money, the less it is worth today.

What is the opportunity cost of capital, 7% or 12%? The answer is 7%: That's the rate of return that your company's shareholders could get by investing on their own at the same level of risk as the proposed project. Here the level of risk is zero. (Remember, we are assuming for now that the future value of the office block is known with certainty.) Your shareholders would vote unanimously for the investment project, because the project offers a safe return of 14% versus a safe return of only 7% in financial markets.

The office-block project is therefore a "go," but how much is it worth and how much will the investment add to your wealth? The project produces a cash flow at the end of one year. To find its present value we discount that cash flow by the opportunity cost of capital:

$$\text{Present value} = \text{PV} = \frac{C_1}{1 + r} = \frac{800{,}000}{1.07} = \$747{,}664$$

Suppose that as soon as you have bought the land and paid for the construction, you decide to sell your project. How much could you sell it for? That is an easy question. If the venture

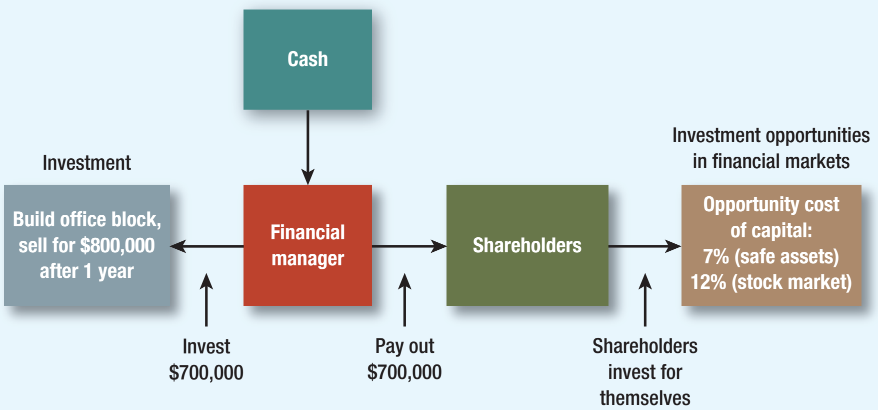


FIGURE 2.3

Your company can either invest $700,000 in an office block and sell it after 1 year for $800,000, or it can return the $700,000 to shareholders to invest in the financial markets.

will return a surefire $800,000, then your property ought to be worth its PV of $747,664 today. That is what investors in the financial markets would need to pay to get the same future payoff. If you tried to sell it for more than $747,664, there would be no takers, because the property would then offer an expected rate of return lower than the 7% available on government securities. Of course, you could always sell your property for less, but why sell for less than the market will bear? The $747,664 present value is the only feasible price that satisfies both buyer and seller. Therefore, the present value of the property is also its market price.

Net Present Value

The office building is worth $747,664 today, but that does not mean you are $747,664 better off. You invested $700,000, so the **net present value (NPV)** is $47,664. Net present value equals present value minus the required investment:

$$\text{NPV} = \text{PV} - \text{investment} = 747{,}664 - 700{,}000 = \$47{,}664$$

In other words, your office development is worth more than it costs. It makes a *net* contribution to value and increases your wealth. The formula for calculating the NPV of your project can be written as:

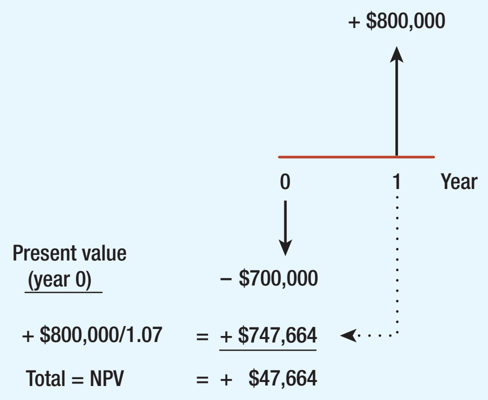
$$\text{NPV} = C_0 + C_1/(1 + r)$$

Remember that C_0, the cash flow at time 0 (that is, today) is usually a negative number. In other words, C_0 is an investment and therefore a cash outflow. In our example, $C_0 = -\$700{,}000$.

When cash flows occur at different points in time, it is often helpful to draw a time line showing the date and value of each cash flow. Figure 2.4 shows a time line for your office development. It sets out the net present value calculation assuming that the discount rate r is 7%.[1]

FIGURE 2.4

Calculation showing the NPV of the office development.



Risk and Present Value

We made one unrealistic assumption in our discussion of the office development: Your real estate adviser cannot be certain about the profitability of an office building. Those future cash flows represent the best forecast, but they are not a sure thing.

If the cash flows are uncertain, your calculation of NPV is wrong. Investors could achieve those cash flows with certainty by buying $747,664 worth of U.S. government securities, so they would not buy your building for that amount. You would have to cut your asking price to attract investors' interest.

Here we can invoke a second basic financial principle: *A safe dollar is worth more than a risky dollar.* Most investors dislike risky ventures and won't invest in them unless they see the prospect of a higher return. However, the concepts of present value and the opportunity cost of capital still make sense for risky investments. It is still proper to discount the payoff by the rate of return offered by a risk-equivalent investment in financial

[1] You sometimes hear lay people refer to "net present value" when they mean "present value," and vice versa. Just remember, *present value* is the value of the investment today; *net present value* is the addition that the investment makes to your wealth.

markets. But we have to think of *expected* payoffs and the *expected* rates of return on other investments.[2]

Not all investments are equally risky. The office development is more risky than a government security but less risky than a start-up biotech venture. Suppose you believe the project is as risky as investment in the stock market and that stocks are expected to provide a 12% return. Then 12% is the opportunity cost of capital for your project. That is what you are giving up by investing in the office building and *not* investing in equally risky securities.

Now recompute NPV with $r = .12$:

$$\text{PV} = \frac{800{,}000}{1.12} = \$714{,}286$$

$$\text{NPV} = \text{PV} - 700{,}000 = \$14{,}286$$

The office building still makes a net contribution to value, but the increase in your wealth is smaller than in our first calculation, which assumed that the cash flows from the project were risk-free.

The value of the office building depends, therefore, on the timing of the cash flows and their risk. The $800,000 payoff would be worth just that if you could get it today. If the office building is as risk-free as government securities, the delay in the cash flow reduces value by $52,336 to $747,664. If the building is as risky as investment in the stock market, then the risk further reduces value by $33,378 to $714,286.

Unfortunately, adjusting asset values for both time and risk is often more complicated than our example suggests. Therefore, we take the two effects separately. For the most part, we dodge the problem of risk in Chapters 2 through 6, either by treating all cash flows as if they were known with certainty or by talking about expected cash flows and expected rates of return without worrying how risk is defined or measured. Then in Chapter 7 we turn to the problem of understanding how financial markets cope with risk.

Present Values and Rates of Return

We have decided that constructing the office building is a smart thing to do, since it is worth more than it costs. To discover how much it is worth, we asked how much you would need to invest directly in securities to achieve the same payoff. That is why we discounted the project's future payoff by the rate of return offered by these equivalent-risk securities—the overall stock market in our example.

We can state our decision rule in another way: your real estate venture is worth undertaking because its rate of return exceeds the opportunity cost of capital. The rate of return is simply the profit as a proportion of the initial outlay:

$$\text{Return} = \frac{\text{profit}}{\text{investment}} = \frac{800{,}000 - 700{,}000}{700{,}000} = .143\text{, or } 14.3\%$$

The cost of capital is once again the return foregone by *not* investing in financial markets. If the office building is as risky as investing in the stock market, the return foregone is 12%. Since the 14.3% return on the office building exceeds the 12% opportunity cost, you should go ahead with the project.

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[2]We define "expected" more carefully in Chapter 9. For now think of expected payoff as a realistic forecast, neither optimistic nor pessimistic. Forecasts of expected payoffs are correct on average.

Building the office block is a smart thing to do, even if the payoff is just as risky as the stock market. We can justify the investment by either one of the following two rules:[3]

- *Net present value rule.* Accept investments that have positive net present values.
- *Rate of return rule.* Accept investments that offer rates of return in excess of their opportunity costs of capital.

Both rules give the same answer, although we will encounter some cases in Chapter 5 where the rate of return rule is unreliable. In those cases, you should use the net present value rule.

Calculating Present Values When There Are Multiple Cash Flows

One of the nice things about present values is that they are all expressed in current dollars—so you can add them up. In other words, the present value of cash flow (A + B) is equal to the present value of cash flow A plus the present value of cash flow B.

Suppose that you wish to value a stream of cash flows extending over a number of years. Our rule for adding present values tells us that the *total* present value is:

$$\text{PV} = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \cdots + \frac{C_T}{(1+r)^T}$$

This is called the **discounted cash flow** (or **DCF**) formula. A shorthand way to write it is

$$\text{PV} = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}$$

where Σ refers to the sum of the series. To find the *net* present value (NPV) we add the (usually negative) initial cash flow:

$$\text{NPV} = C_0 + \text{PV} = C_0 + \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}$$

EXAMPLE 2.1 ● Present Values with Multiple Cash Flows

Your real estate adviser has come back with some revised forecasts. He suggests that you rent out the building for two years at \$30,000 a year, and predicts that at the end of that time you will be able to sell the building for \$840,000. Thus there are now two future cash flows—a cash flow of $C_1 = \$30{,}000$ at the end of one year and a further cash flow of $C_2 = (30{,}000 + 840{,}000) = \$870{,}000$ at the end of the second year.

The present value of your property development is equal to the present value of C_1 plus the present value of C_2. Figure 2.5 shows that the value of the first year's cash flow is $C_1/(1 + r) = 30{,}000/1.12 = \$26{,}786$ and the value of the second year's flow is $C_2/(1 + r)^2 = 870{,}000/1.12^2 = \$693{,}559$. Therefore our rule for adding present values tells us that the *total* present value of your investment is:

$$\text{PV} = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} = \frac{30{,}000}{1.12} + \frac{870{,}000}{1.12^2} = 26{,}786 + 693{,}559 = \$720{,}344$$

[3] You might check for yourself that these are equivalent rules. In other words, if the return of \$100,000/\$700,000 is greater than *r*, then the net present value $-\$700{,}000 + [\$800{,}000/(1 + r)]$ *must* be greater than 0.

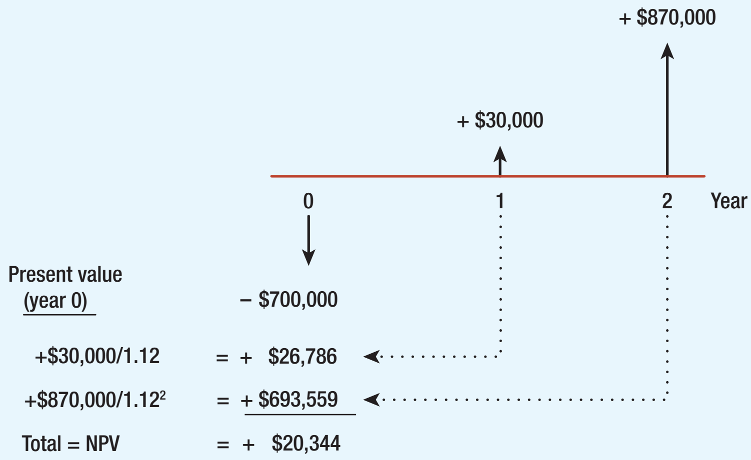

FIGURE 2.5

Calculation showing the NPV of the revised office project.

It looks as if you should take your adviser's suggestion. NPV is higher than if you sell in year 1:

$$\text{NPV} = \$720{,}344 - \$700{,}000 = \$20{,}344$$

Your two-period calculations in Example 2.1 required just a few keystrokes on a calculator. Real problems can be much more complicated, so financial managers usually turn to financial calculators especially programmed for present value calculations or to computer spreadsheet programs. A box near the end of the chapter introduces you to some useful Excel functions that can be used to solve discounting problems.

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Introduction to financial calculators

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The Opportunity Cost of Capital

By investing in the office building you are giving up the opportunity to earn an expected return of 12% in the stock market. The opportunity cost of capital is therefore 12%. When you discount the expected cash flows by the opportunity cost of capital, you are asking how much investors in the financial markets are prepared to pay for a security that produces a similar stream of future cash flows. Your calculations showed that these investors would need to pay $720,344 for an investment that produces cash flows of $30,000 at year 1 and $870,000 at year 2. Therefore, they won't pay any more than that for your office building.

Confusion sometimes sneaks into discussions of the cost of capital. Suppose a banker approaches. "Your company is a fine and safe business with few debts," she says. "My bank will lend you the $700,000 that you need for the office block at 8%." Does this mean that the cost of capital is 8%? If so, the project would be even more worthwhile. At an 8% cost of capital, PV would be $30{,}000/1.08 + 870{,}000/1.08^2 = \$773{,}663$ and $\text{NPV} = \$773{,}663 - \$700{,}000 = +\$73{,}663$.

But that can't be right. First, the interest rate on the loan has nothing to do with the risk of the project: it reflects the good health of your existing business. Second, whether you take the loan or not, you still face the choice between the office building and an equally risky investment in the stock market. The stock market investment could generate the same expected payoff as your office building at a lower cost. A financial manager who borrows $700,000 at 8% and invests in an office building is not smart, but stupid, if the company or its shareholders can borrow at 8% and invest the money at an even higher return. That is why the 12% expected return on the stock market is the opportunity cost of capital for your project.

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Introduction to Excel

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2-2 Looking for Shortcuts—Perpetuities and Annuities

How to Value Perpetuities

Sometimes there are shortcuts that make it easy to calculate present values. Let us look at some examples.

On occasion, the British and the French have been known to disagree and sometimes even to fight wars. At the end of some of these wars the British consolidated the debt they had issued during the war. The securities issued in such cases were called consols. Consols are **perpetuities.** These are bonds that the government is under no obligation to repay but that offer a fixed income for each year to perpetuity. The British government is still paying interest on consols issued all those years ago. The annual rate of return on a perpetuity is equal to the promised annual payment divided by the present value:[4]

$$\text{Return} = \frac{\text{cash flow}}{\text{present value}}$$

$$r = \frac{C}{\text{PV}}$$

We can obviously twist this around and find the present value of a perpetuity given the discount rate r and the cash payment C:

$$\text{PV} = \frac{C}{r}$$

The year is 2030. You have been fabulously successful and are now a billionaire many times over. It was fortunate indeed that you took that finance course all those years ago. You have decided to follow in the footsteps of two of your heroes, Bill Gates and Warren Buffet. Malaria is still a scourge and you want to help eradicate it and other infectious diseases by endowing a foundation to combat these diseases. You aim to provide \$1 billion a year in perpetuity, starting next year. So, if the interest rate is 10%, you are going to have to write a check today for

$$\text{Present value of perpetuity} = \frac{C}{r} = \frac{\$1 \text{ billion}}{.1} = \$10 \text{ billion}$$

Two warnings about the perpetuity formula. First, at a quick glance you can easily confuse the formula with the present value of a single payment. A payment of \$1 at the end of one year has a present value of $1/(1 + r)$. The perpetuity has a value of $1/r$. These are quite different.

Second, the perpetuity formula tells us the value of a regular stream of payments starting one period from now. Thus your \$10 billion endowment would provide the foundation with its first payment in one year's time. If you also want to provide an up-front sum, you will need to lay out an extra \$1 billion.

Sometimes you may need to calculate the value of a perpetuity that does not start to make payments for several years. For example, suppose that you decide to provide \$1 billion a year with the first payment four years from now. Figure 2.6 provides a timeline of these payments.

[4]You can check this by writing down the present value formula

$$\text{PV} = \frac{C}{1 + r} + \frac{C}{(1 + r)^2} + \frac{C}{(1 + r)^3} + \cdots$$

Now let $C/(1 + r) = a$ and $1/(1 + r) = x$. Then we have (1) $\text{PV} = a(1 + x + x^2 + \ldots)$. Multiplying both sides by x, we have (2) $\text{PV}x = a(x + x^2 + \ldots)$. Subtracting (2) from (1) gives us $\text{PV}(1 - x) = a$. Therefore, substituting for a and x,

$$\text{PV}\left(1 - \frac{1}{1 + r}\right) = \frac{C}{1 + r}$$

Multiplying both sides by $(1 + r)$ and rearranging gives

$$\text{PV} = \frac{C}{r}$$

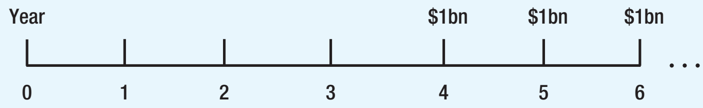

FIGURE 2.6
This perpetuity makes a series of payments of $1 billion a year starting in year 4.

Think first about how much they will be worth in year 3. At that point the endowment will be an ordinary perpetuity with the first payment due at the end of the year. So our perpetuity formula tells us that in year 3 the endowment will be worth $\$1/r = \$1/.1 = \$10$ billion. But it is not worth that much now. To find *today's* value we need to multiply by the three-year discount factor $1/(1 + r)^3 = 1/(1.1)^3 = .751$. Thus, the "delayed" perpetuity is worth $10 billion × .751 = $7.51 billion. The full calculation is:

$$\text{PV} = \$1 \text{ billion} \times \frac{1}{r} \times \frac{1}{(1 + r)^3} = \$1 \text{ billion} \times \frac{1}{.10} \times \frac{1}{(1.10)^3} = \$7.51 \text{ billion}$$

How to Value Annuities

An **annuity** is an asset that pays a fixed sum each year for a specified number of years. The equal-payment house mortgage or installment credit agreement are common examples of annuities. So are interest payments on most bonds, as we see in the next chapter.

You can always value an annuity by calculating the value of each cash flow and finding the total. However, it is often quicker to use a simple formula that states that if the interest rate is r, then the present value of an annuity that pays $\$C$ a period for each of t periods is:

$$\text{Present value of } t\text{-year annuity} = C\left[\frac{1}{r} - \frac{1}{r(1 + r)^t}\right]$$

The expression in brackets shows the present value of $1 a year for each of t years. It is generally known as the t-year **annuity factor.**

If you are wondering where this formula comes from, look at Figure 2.7. It shows the payments and values of three investments.

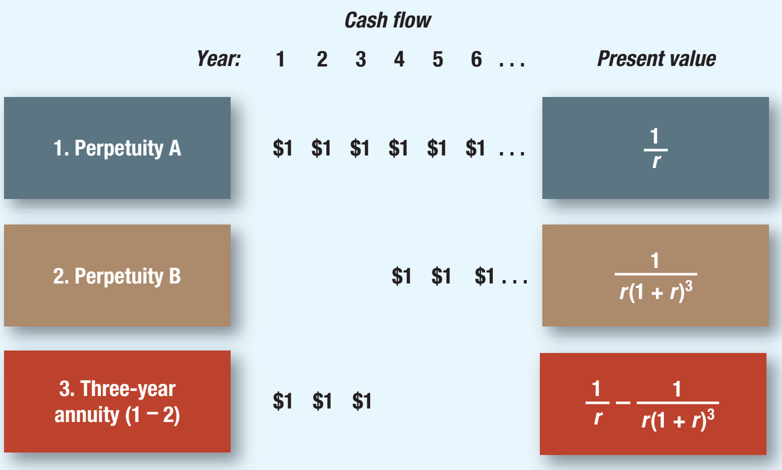

FIGURE 2.7
An annuity that makes payments in each of years 1 through 3 is equal to the difference between two perpetuities.

Row 1 The investment in the first row provides a perpetual stream of \$1 starting at the end of the first year. We have already seen that this perpetuity has a present value of $1/r$.

Row 2 Now look at the investment shown in the second row of Figure 2.7. It also provides a perpetual stream of \$1 payments, but these payments don't start until year 4. This stream of payments is identical to the payments in row 1, except that they are delayed for an additional three years. In year 3, the investment will be an ordinary perpetuity with payments starting in one year and will therefore be worth $1/r$ in year 3. To find the value *today,* we simply multiply this figure by the three-year discount factor. Thus

$$\text{PV} = \frac{1}{r} \times \frac{1}{(1 + r)^3}$$

Row 3 Finally, look at the investment shown in the third row of Figure 2.7. This provides a level payment of \$1 a year for each of three years. In other words, it is a three-year annuity. You can also see that, taken together, the investments in rows 2 and 3 provide exactly the same cash payments as the investment in row 1. Thus the value of our annuity (row 3) must be equal to the value of the row 1 perpetuity less the value of the delayed row 2 perpetuity:

$$\text{Present value of a 3-year annuity of \$1 a year} = \frac{1}{r} - \frac{1}{r(1 + r)^3}$$

Remembering formulas is about as difficult as remembering other people's birthdays. But as long as you bear in mind that an annuity is equivalent to the difference between an immediate and a delayed perpetuity, you shouldn't have any difficulty.[5]

EXAMPLE 2.2 • Costing an Installment Plan

Most installment plans call for level streams of payments. Suppose that Tiburon Autos offers an "easy payment" scheme on a new Toyota of \$5,000 a year, paid at the end of each of the next five years, with no cash down. What is the car really costing you?

First let us do the calculations the slow way, to show that, if the interest rate is 7%, the present value of these payments is \$20,501. The time line in Figure 2.8 shows the value of each cash flow and the total present value. The annuity formula, however, is generally quicker; you simply need to multiply the \$5,000 cash flow by the annuity factor:

$$\text{PV} = 5{,}000\left[\frac{1}{.07} - \frac{1}{.07(1.07)^5}\right] = 5{,}000 \times 4.100 = \$20{,}501$$

[5]Some people find the following equivalent formula more intuitive:

$$\text{Present value of annuity} = \underbrace{\frac{1}{r}}_{\substack{\text{perpetuity}\\\text{formula}}} \times \left[\underbrace{1}_{\substack{\text{\$1}\\\text{starting}\\\text{next year}}} - \underbrace{\frac{1}{(1 + r)^t}}_{\substack{\text{minus \$1}\\\text{starting at}\\ t+1}}\right]$$

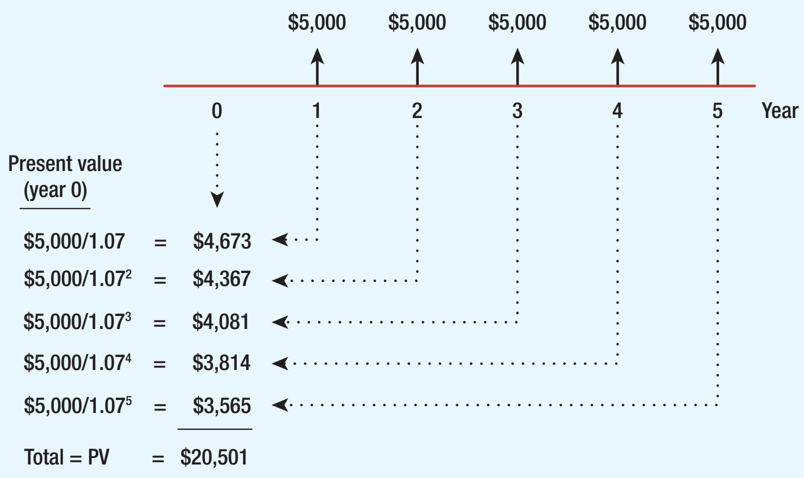

FIGURE 2.8
Calculations showing the year-by-year present value of the installment payments.

EXAMPLE 2.3 ● Winning Big at the Lottery

In August 2006 eight lucky meatpackers from Nebraska pooled their money to buy Powerball lottery tickets and won a record $365 million. We suspect that the winners received unsolicited congratulations, good wishes, and requests for money from dozens of more or less worthy charities, relations, and newly devoted friends. In response, they could fairly point out that the prize wasn't really worth $365 million. That sum was to be paid in 30 equal annual installments of $12.167 million each. Assuming that the first payment occurred at the end of one year, what was the present value of the prize? The interest rate at the time was 6.0%.

These payments constitute a 30-year annuity. To value this annuity we simply multiply $12.167 million by the 30-year annuity factor:

$$\begin{aligned} \text{PV} &= 12.167 \times \text{30-year annuity factor} \\ &= 12.167 \times \left[\frac{1}{r} - \frac{1}{r(1+r)^{30}}\right] \end{aligned}$$

At an interest rate of 6.0%, the annuity factor is

$$\left[\frac{1}{.060} - \frac{1}{.060(1.060)^{30}}\right] = 13.765$$

The present value of the cash payments is $12.167 × 13.765 = $167.5 million, much below the well-trumpeted prize, but still not a bad day's haul.

Lottery operators generally make arrangements for winners with big spending plans to take an equivalent lump sum. In our example the winners could either take the $365 million spread over 30 years or receive $167.5 million up front. Both arrangements had the same present value.

Valuing Annuities Due

When we used the annuity formula to value the Powerball lottery prize in Example 2.3, we presupposed that the first payment was made at the end of one year. In fact, the first of the 30 yearly payments was made immediately. How does this change the value of the prize?

If we discount each cash flow by one less year, the present value is increased by the multiple $(1 + r)$. In the case of the lottery prize the value becomes $167.5 \times (1 + r) = 167.5 \times 1.060 =$ \$177.5 million.

A level stream of payments starting immediately is called an **annuity due.** An annuity due is worth $(1 + r)$ times the value of an ordinary annuity.

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Try It! More on annuities

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Calculating Annual Payments

Annuity problems can be confusing on first acquaintance, but you will find that with practice they are generally straightforward. For example, here is a case where you need to use the annuity formula to find the amount of the payment *given* the present value.

EXAMPLE 2.4 ● Paying Off a Bank Loan

Bank loans are paid off in equal installments. Suppose that you take out a four-year loan of \$1,000. The bank requires you to repay the loan evenly over the four years. It must therefore set the four annual payments so that they have a present value of \$1,000. Thus,

$$\text{PV} = \text{annual loan payment} \times \text{4-year annuity factor} = \$1{,}000$$

$$\text{Annual loan payment} = \$1{,}000/\text{4-year annuity factor}$$

Suppose that the interest rate is 10% a year. Then

$$\text{4-year annuity factor} = \left[\frac{1}{.10} - \frac{1}{.10(1.10)^4}\right] = 3.17$$

and

$$\text{Annual loan payment} = 1{,}000/3.17 = \$315.47$$

Let's check that this annual payment is sufficient to repay the loan. Table 2.1 provides the calculations. At the end of the first year, the interest charge is 10% of \$1,000, or \$100. So \$100 of the first payment is absorbed by interest, and the remaining \$215.47 is used to reduce the loan balance to \$784.53.

Next year, the outstanding balance is lower, so the interest charge is only \$78.45. Therefore \$315.47 − \$78.45 = \$237.02 can be applied to paying off the loan. Because the loan is

Year	Beginning-of-Year Balance	Year-end Interest on Balance	Total Year-end Payment	Amortization of Loan	End-of-Year Balance
1	\$1,000.00	\$100.00	\$315.47	\$215.47	\$784.53
2	784.53	78.45	315.47	237.02	547.51
3	547.51	54.75	315.47	260.72	286.79
4	286.79	28.68	315.47	286.79	0

TABLE 2.1 An example of an amortizing loan. If you borrow \$1,000 at an interest rate of 10%, you would need to make an annual payment of \$315.47 over four years to repay that loan with interest.

progressively paid off, the fraction of each payment devoted to interest steadily falls over time, while the fraction used to reduce the loan increases. By the end of year 4, the amortization is just enough to reduce the balance of the loan to zero.

Loans that involve a series of level payments are known as *amortizing loans.* "Amortizing" means that part of the regular payment is used to pay interest on the loan and part is used to reduce the amount of the loan.

EXAMPLE 2.5 ● Calculating Mortgage Payments

Most mortgages are amortizing loans. For example, suppose that you take out a $250,000 house mortgage from your local savings bank when the interest rate is 12%. The bank requires you to repay the mortgage in equal annual installments over the next 30 years. Thus,

$$\text{Annual mortgage payment} = \$250{,}000/\text{30-year annuity factor}$$

$$\text{30-year annuity factor} = \left[\frac{1}{.12} - \frac{1}{.12(1.12)^{30}}\right] = 8.055$$

and

$$\text{Annual mortgage payment} = 250{,}000/8.055 = \$31{,}036$$

Figure 2.9 shows that in the early years, almost all of the mortgage payment is eaten up by interest and only a small fraction is used to reduce the amount of the loan. Even after 15 years, the bulk of the annual payment goes to pay the interest on the loan. From then on, the amount of the loan begins to decline rapidly.

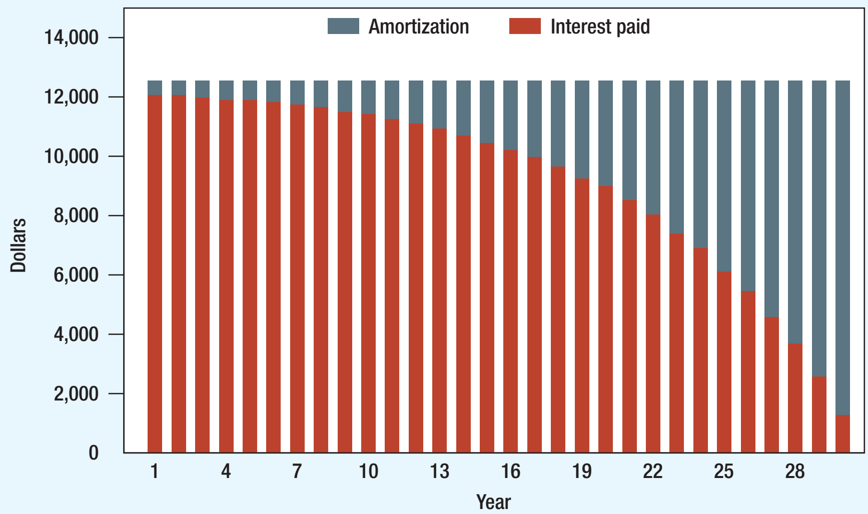

FIGURE 2.9

Mortgage amortization. This figure shows the breakdown of mortgage payments between interest and amortization.

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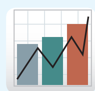
Try It! Figure 2.9: The amortzation schedule



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Future Value of an Annuity

Sometimes you need to calculate the *future* value of a level stream of payments.

EXAMPLE 2.6 ● Saving to Buy a Sailboat

Perhaps your ambition is to buy a sailboat; something like a 40-foot Beneteau would fit the bill very well. But that means some serious saving. You estimate that, once you start work, you could save \$20,000 a year out of your income and earn a return of 8% on these savings. How much will you be able to spend after five years?

We are looking here at a level stream of cash flows—an annuity. We have seen that there is a shortcut formula to calculate the *present* value of an annuity. So there ought to be a similar formula for calculating the *future* value of a level stream of cash flows.

Think first how much your savings are worth today. You will set aside \$20,000 in each of the next five years. The present value of this five-year annuity is therefore equal to

$$\text{PV} = \$20{,}000 \times \text{5-year annuity factor}$$

$$= \$20{,}000 \times \left[\frac{1}{.08} - \frac{1}{.08(1.08)^5}\right] = \$79{,}854$$

Once you know today's value of the stream of cash flows, it is easy to work out its value in the future. Just multiply by $(1.08)^5$:

$$\text{Value at end of year 5} = \$79{,}854 \times 1.08^5 = \$117{,}332$$

You should be able to buy yourself a nice boat for \$117,000.

In Example 2.6 we calculate the future value of an annuity by first calculating its present value and then multiplying by $(1 + r)^t$. The general formula for the future value of a level stream of cash flows of \$1 a year for t years is, therefore,

$$\text{Future value of annuity} = \text{present value of annuity of \$1 a year} \times (1 + r)^t$$

$$= \left[\frac{1}{r} - \frac{1}{r(1 + r)^t}\right] \times (1 + r)^t = \frac{(1 + r)^t - 1}{r}$$

There is a general point here. If you can find the present value of *any* series of cash flows, you can always calculate future value by multiplying by $(1 + r)^t$:

$$\text{Future value at the end of year } t = \text{present value} \times (1 + r)^t$$

2-3 More Shortcuts—Growing Perpetuities and Annuities

Growing Perpetuities

You now know how to value level streams of cash flows, but you often need to value a stream of cash flows that grows at a constant rate. For example, think back to your plans to donate \$10 billion to fight malaria and other infectious diseases. Unfortunately, you made no allowance for the growth in salaries and other costs, which will probably average about 4% a year starting in year 1. Therefore, instead of providing \$1 billion a year in perpetuity, you must provide \$1 billion in year 1, 1.04 × \$1 billion in year 2, and so on. If we call the growth rate in costs *g,* we can write down the present value of this stream of cash flows as follows:

$$\mathrm{PV} = \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \cdots$$

$$= \frac{C_1}{1+r} + \frac{C_1(1+g)}{(1+r)^2} + \frac{C_1(1+g)^2}{(1+r)^3} + \cdots$$

Fortunately, there is a simple formula for the sum of this geometric series.[6] If we assume that r is greater than g, our clumsy-looking calculation simplifies to

$$\text{Present value of growing perpetuity} = \frac{C_1}{r-g}$$

Therefore, if you want to provide a perpetual stream of income that keeps pace with the growth rate in costs, the amount that you must set aside today is

$$\mathrm{PV} = \frac{C_1}{r-g} = \frac{\$1 \text{ billion}}{.10 - .04} = \$16.667 \text{ billion}$$

You will meet this perpetual-growth formula again in Chapter 4, where we use it to value the stock of mature, slowly growing companies.

Growing Annuities

You are contemplating membership in the St. Swithin's and Ancient Golf Club. The annual membership fee for the coming year is \$5,000, but you can make a single payment today of \$12,750, which will provide you with membership for the next three years. Which is the better deal? The answer depends on how rapidly membership fees are likely to increase over the three-year period. For example, suppose that the annual fee is payable at the end of each year and is expected to increase by 6% per annum. The discount rate is 10%.

The problem is to calculate the present value of the three-year stream of growing payments. The first payment occurs at the end of year 1 and is $C = \$5{,}000$. Thereafter, the payments grow at the rate of $g = .06$ each year. Thus in year 2 the expected payment is \$5,000 × 1.06, and in

[6]We need to calculate the sum of an infinite geometric series $\mathrm{PV} = a(1 + x + x^2 + \cdots)$ where $a = C_1/(1+r)$ and $x = (1+g)/(1+r)$. In footnote 4 we showed that the sum of such a series is $a/(1-x)$. Substituting for a and x in this formula,

$$\mathrm{PV} = \frac{C_1}{r-g}$$

				Cash Flow, $			
Year:	**0**	**1**	**2 . . .**	**. . . t − 1**	**t**	**t + 1 . . .**	**Present Value**
Perpetuity		1	1 . . .	1	1	1 . . .	$\frac{1}{r}$
t-period annuity		1	1 . . .	1	1		$\frac{1}{r} - \frac{1}{r(1+r)^t}$
t-period annuity due	1	1	1 . . .	1			$(1+r)\left(\frac{1}{r} - \frac{1}{r(1+r)^t}\right)$
Growing perpetuity		1	$1 \times (1+g)$ . . .	$1 \times (1+g)^{t-2}$	$1 \times (1+g)^{t-1}$	$1 \times (1+g)^{t}$ . .	$\frac{1}{r-g}$
t-period growing annuity		1	$1 \times (1+g)$ . . .	$1 \times (1+g)^{t-2}$	$1 \times (1+g)^{t-1}$		$\frac{1}{r-g}\left[1 - \frac{(1+g)^t}{(1+r)^t}\right]$

TABLE 2.2 Some useful shortcut formulas.

year 3 it is $\$5{,}000 \times 1.06^2$. Of course, you could calculate these cash flows and discount them at 10%. The alternative is to use the following formula for the present value of a growing annuity:[7]

$$\text{PV of growing annuity} = C \times \frac{1}{r-g}\left[1 - \frac{(1+g)^t}{(1+r)^t}\right]$$

In our golf club example, the present value of the membership fees for the next three years is

$$\text{PV} = \$5{,}000 \times \frac{.1}{.10 - .06}\left[1 - \frac{(1.06)^3}{(1.10)^3}\right] = \$5{,}000 \times 2.629 = \$13{,}147$$

If you can find the cash, you would be better off paying now for a three-year membership.

Too many formulas are bad for the digestion. So we will stop at this point and spare you any more of them. The formulas discussed so far appear in Table 2.2.

[7]We can derive the formula for a growing perpetuity by taking advantage of our earlier trick of finding the difference between the values of two perpetuities. Imagine three investments (A, B, and C) that make the following dollar payments:

Year	1	2	3	4	5	6	. . .
A	\$1	$(1+g)$	$(1+g)^2$	$(1+g)^3$	$(1+g)^4$	$(1+g)^5$	etc.
B				$(1+g)^3$	$(1+g)^4$	$(1+g)^5$	etc.
C	\$1	$(1+g)$	$(1+g)^2$				

Investments A and B are growing perpetuities; A makes its first payment of \$1 in year 1, while B makes its first payment of $\$(1+g)^3$ in year 4. C is a three-year growing annuity; its cash flows are equal to the difference between the cash flows of A and B. You know how to value growing perpetuities such as A and B. So you should be able to derive the formula for the value of growing annuities such as C:

$$\text{PV(A)} = \frac{1}{(r-g)}$$

$$\text{PV(B)} = \frac{(1+g)^3}{(r-g)} \times \frac{1}{(1+r)^3}$$

So

$$\text{PV(C)} = \text{PV(A)} - \text{PV(B)} = \frac{1}{(r-g)} - \frac{(1+g)^3}{(r-g)} \times \frac{1}{(1+r)^3} = \frac{1}{r-g}\left[1 - \frac{(1+g)^3}{(1+r)^3}\right]$$

If $r = g$, then the formula blows up. In that case, the cash flows grow at the same rate as the amount by which they are discounted. Therefore, each cash flow has a present value of $C/(1+r)$ and the total present value of the annuity equals $t \times C/(1+r)$. If $r < g$, then this particular formula remains valid.

2-4 How Interest Is Paid and Quoted

In our examples we have assumed that cash flows occur only at the end of each year. This is sometimes the case. For example, in France and Germany the government pays interest on its bonds annually. However, in the United States and Britain government bonds pay interest semiannually. So if the interest rate on a U.S. government bond is quoted as 10%, the investor in practice receives interest of 5% every six months.

If the first interest payment is made at the end of six months, you can earn an additional six months' interest on this payment. For example, if you invest \$100 in a bond that pays interest of 10% compounded semiannually, your wealth will grow to $1.05 \times \$100 = \$105$ by the end of six months and to $1.05 \times \$105 = \$110.25$ by the end of the year. In other words, an interest rate of 10% compounded semiannually is equivalent to 10.25% compounded annually. The *effective annual interest rate* on the bond is 10.25%.

Let's take another example. Suppose a bank offers you an automobile loan at an **annual percentage rate,** or **APR,** of 12% with interest to be paid monthly. This means that each month you need to pay one-twelfth of the annual rate, that is, $12/12 = 1\%$ a month. Thus the bank is *quoting* a rate of 12%, but the effective annual interest rate on your loan is $1.01^{12} - 1 = .1268$, or 12.68%.[8]

Our examples illustrate that you need to distinguish between the *quoted* annual interest rate and the *effective* annual rate. The quoted annual rate is usually calculated as the total annual payment divided by the number of payments in the year. When interest is paid once a year, the quoted and effective rates are the same. When interest is paid more frequently, the effective interest rate is higher than the quoted rate.

In general, if you invest \$1 at a rate of r per year compounded m times a year, your investment at the end of the year will be worth $[1 + (r/m)]^m$ and the effective interest rate is $[1 + (r/m)]^m - 1$. In our automobile loan example $r = .12$ and $m = 12$. So the effective annual interest rate was $[1 + .12/12]^{12} - 1 = .1268$, or 12.68%.

Continuous Compounding

Instead of compounding interest monthly or semiannually, the rate could be compounded weekly ($m = 52$) or daily ($m = 365$). In fact there is no limit to how frequently interest could be paid. One can imagine a situation where the payments are spread evenly and continuously throughout the year, so the interest rate is continuously compounded.[9] In this case m is infinite.

It turns out that there are many occasions in finance when continuous compounding is useful. For example, one important application is in option pricing models, such as the Black–Scholes model that we introduce in Chapter 21. These are continuous time models. So you will find that most computer programs for calculating option values ask for the continuously compounded interest rate.

It may seem that a lot of calculations would be needed to find a continuously compounded interest rate. However, think back to your high school algebra. You may recall that as m approaches infinity $[1 + (r/m)]^m$ approaches $(2.718)^r$. The figure 2.718—or *e*, as it is called—is the base for natural logarithms. Therefore, \$1 invested at a continuously compounded rate of r will grow to $e^r = (2.718)^r$ by the end of the first year. By the end of t years it will grow to $e^{rt} = (2.718)^{rt}$.

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[8]In the U.S., truth-in-lending laws oblige the company to quote an APR that is calculated by multiplying the payment each period by the number of payments in the year. APRs are calculated differently in other countries. For example, in the European Union APRs must be expressed as annually compounded rates, so consumers know the effective interest rate that they are paying.

[9]When we talk about *continuous* payments, we are pretending that money can be dispensed in a continuous stream like water out of a faucet. One can never quite do this. For example, instead of paying out \$1 billion every year to combat malaria, you could pay out about \$1 million every 8¾ hours or \$10,000 every 5¼ minutes or \$10 every $3^1/_6$ seconds but you could not pay it out *continuously.* Financial managers *pretend* that payments are continuous rather than hourly, daily, or weekly because (1) it simplifies the calculations and (2) it gives a very close approximation to the NPV of frequent payments.

Example 1 Suppose you invest \$1 at a continuously compounded rate of 11% ($r = .11$) for one year ($t = 1$). The end-year value is $e^{.11}$, or \$1.116. In other words, investing at 11% a year *continuously* compounded is exactly the same as investing at 11.6% a year *annually* compounded.

Example 2 Suppose you invest \$1 at a continuously compounded rate of 11% ($r = .11$) for two years ($t = 2$). The final value of the investment is $e^{rt} = e^{.22}$, or \$1.246.

Sometimes it may be more reasonable to assume that the cash flows from a project are spread evenly over the year rather than occurring at the year's end. It is easy to adapt our previous formulas to handle this. For example, suppose that we wish to compute the present value of a perpetuity of C dollars a year. We already know that if the payment is made at the end of the year, we divide the payment by the *annually* compounded rate of r:

$$\text{PV} = \frac{C}{r}$$

If the same total payment is made in an even stream throughout the year, we use the same formula but substitute the *continuously* compounded rate.

Suppose the annually compounded rate is 18.5%. The present value of a \$100 perpetuity, with each cash flow received at the end of the year, is $100/.185 = \$540.54$. If the cash flow is received continuously, we must divide \$100 by 17%, because 17% continuously compounded is equivalent to 18.5% annually compounded ($e^{.17} = 1.185$). The present value of the continuous cash flow stream is $100/.17 = \$588.24$. Investors are prepared to pay more for the continuous cash payments because the cash starts to flow in immediately.

Example 3 After you have retired, you plan to spend \$200,000 a year for 20 years. The annually compounded interest rate is 10%. How much must you save by the time you retire to support this spending plan?

Let us first do the calculations assuming that you spend the cash at the end of each year. In this case we can use the simple annuity formula that we derived earlier:

$$\begin{aligned}\text{PV} &= C\left(\frac{1}{r} - \frac{1}{r} \times \frac{1}{(1+r)^t}\right)\\ &= \$200{,}000\left(\frac{1}{.10} - \frac{1}{.10} \times \frac{1}{(1.10)^{20}}\right) = \$200{,}000 \times 8.514 = \$1{,}702{,}800\end{aligned}$$

Thus, you will need to have saved nearly \$1¾ million by the time you retire.

Instead of waiting until the end of each year before you spend any cash, it is more reasonable to assume that your expenditure will be spread evenly over the year. In this case, instead of using the annually compounded rate of 10%, we must use the continuously compounded rate of $r = 9.53\%$ ($e^{.0953} = 1.10$). Therefore, to cover a steady stream of expenditure, you need to set aside the following sum:[10]

[10]Remember that an annuity is simply the difference between a perpetuity received today and a perpetuity received in year t. A continuous stream of C dollars a year in perpetuity is worth C/r, where r is the continuously compounded rate. Our annuity, then, is worth

$$\text{PV} = \frac{C}{r} - \text{present value of } \frac{C}{r} \text{ received in year } t$$

Since r is the continuously compounded rate, C/r received in year t is worth $(C/r) \times (1/e^{rt})$ today. Our annuity formula is therefore

$$\text{PV} = \frac{C}{r} - \frac{C}{r} \times \frac{1}{e^{rt}}$$

sometimes written as

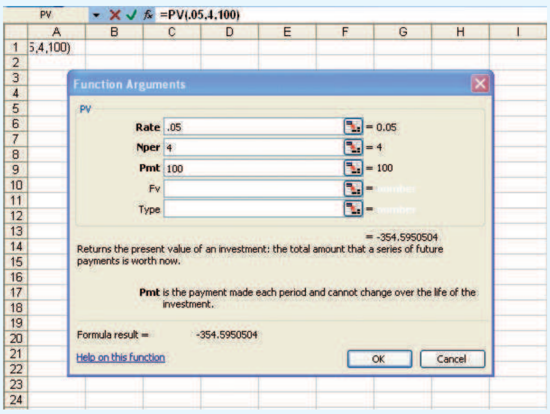
$$\frac{C}{r}(1 - e^{-rt})$$

Discounting Cash Flows

Spreadsheet programs such as Excel provide built-in functions to solve discounted-cash-flow (DCF) problems. You can find these functions by pressing *fx* on the Excel toolbar. If you then click on the function that you wish to use, Excel asks you for the inputs that it needs. At the bottom left of the function box there is a Help facility with an example of how the function is used.

Here is a list of useful functions for DCF problems and some points to remember when entering data:

- **FV:** Future value of single investment or annuity.
- **PV:** Present value of single future cash flow or annuity.
- **RATE:** Interest rate (or rate of return) needed to produce given future value or annuity.
- **NPER:** Number of periods (e.g., years) that it takes an investment to reach a given future value or series of future cash flows.
- **PMT:** Amount of annuity payment with a given present or future value.
- **NPV:** Calculates the value of a stream of negative and positive cash flows. (When using this function, note the warning below.)
- **XNPV:** Calculates the net present value at the date of the first cash flow of a series of cash flows occurring at uneven intervals.
- **EFFECT:** The effective annual interest rate, given the quoted rate (APR) and number of interest payments in a year.



- **NOMINAL:** The quoted interest rate (APR) given the effective annual interest rate.

All the inputs in these functions can be entered directly as numbers or as the addresses of cells that contain the numbers.

Three warnings:

1. PV is the amount that needs to be invested today to produce a given future value. It should therefore be entered as a negative number. Entering both PV and FV with the same sign when solving for RATE results in an error message.
2. Always enter the interest or discount rate as a decimal value (e.g., .05 rather than 5%).
3. Use the NPV function with care. Better still, don't use it at all. It gives the value of the cash flows one period *before* the first cash flow and not the value at the date of the first cash flow.

Spreadsheet Questions

The following questions provide opportunities to practice each of the Excel functions.

1. (FV) In 1880 five aboriginal trackers were each promised the equivalent of 100 Australian dollars for helping to capture the notorious outlaw Ned Kelly. One hundred and thirteen years later the granddaughters of two of the trackers claimed that this reward had not been paid. If the interest rate over this period averaged about 4.5%, how much would the A$100 have accumulated to?
2. (PV) Your company can lease a truck for $10,000 a year (paid at the end of the year) for six years, or it can buy the truck today for $50,000. At the end of the six years the truck will be worthless. If the interest rate is 6%, what is the present value of the lease payments? Is the lease worthwhile?
3. (RATE) Ford Motor stock was one of the victims of the 2008 credit crisis. In June 2007, Ford stock price stood at $9.42. Eighteen months later it was $2.72. What was the annual rate of return over this period to an investor in Ford stock?
4. (NPER) An investment adviser has promised to double your money. If the interest rate is 7% a year, how many years will she take to do so?
5. (PMT) You need to take out a home mortgage for $200,000. If payments are made annually over 30 years and the interest rate is 8%, what is the amount of the annual payment?